Zeros of the zeta function and Mellin transform, some formulas

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ZEROS OF THE ZETA FUNCTION AND MELLIN TRANSFORM, SOME FORMULAS

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Abstract

In this paper, we prove a formula, expressing, in terms of the psi function and of the Riemann zeta function, the non-trivial zeros of the Riemann zeta function, and, more generally, any analytic function of these zeros. Two methods are used, one of them using the Mellin transform.

Theorem.

If f is holomorphic on a neighborhood of the half-band $\left\{-\frac{1}{2} \leq \Re(s) \leq \frac{1}{2}, \Im(s) \leq 0\right\}$, if $\left(\frac{1}{2}+ib_k\right)_{k\in\mathbb{Z}-\{0\}}$ is the sequence of the zeros of the zeta function having real part $\frac{1}{2}$, if $(c_k+id_k)_{k\in\mathbb{Z}-\{0\}}$ is the sequence of the remaining non-trivial zeros (with the following conventions: $b_k, d_k > 0$ if $k > 0$, $0 < c_k < 1$, $c_k \neq \frac{1}{2}$, $b_{-k} = -b_k$, $c_{-k} + id_{-k} = 1 - (c_k + id_k)$ if $k \in \mathbb{Z} - \{0\}$), and if n and A are two real numbers such that $n > 0$, $n \notin \{b_k, d_k, k > 0\}$ and $A > \frac{1}{2}$, then we have:

$$\begin{aligned}
&2\pi \sum_{0<b_k<n} f(-ib_k) + 2\pi \sum_{0<d_k<n} f\left(\frac{1}{2} - c_k - id_k\right) = \\
&\frac{\ln(\pi)}{i} \int_0^{-in} f(s)\,\mathrm{d}s - \frac{1}{2i}\int_0^{-in} f(s)\left(\Psi\left(\frac{1}{4}-\frac{s}{2}\right)+\Psi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\mathrm{d}s \qquad (1)\\
&-\frac{1}{i}\left(\int_{R^*_{n,A}} f(s)\,\frac{\zeta'}{\zeta}\left(\frac{1}{2}+s\right)\mathrm{d}s + \int_{R^*_{n,-A}} f(s)\,\frac{\zeta'}{\zeta}\left(\frac{1}{2}-s\right)\mathrm{d}s\right),
\end{aligned}$$

where Ψ is the digamma function, ζ is the Riemann zeta function , $\int_0^{-in}$ refers to the integral along the oriented segment $[0, -in]$ and where the path of integration $R^*_{n,A}$ (resp. $R^*_{n,-A}$) is the portion of the rectangle in U successively joining the points 0, A (resp. $-A$), $A - in$ (resp. $-A - in$), $-in$ (with a small semicircle below $\frac{1}{2}$ (resp. $-\frac{1}{2}$) .

Remark.

a. If $A < 0$ (and if we avoid the poles of zeta), the first member of the formula (1) is changed into its opposite.

b. The path of integration $R^*_{n,A}$ (resp. $R^*_{n,-A}$) can be replaced by any homotopic path in $U \cap \mathbb{C} - \{\text{poles of zeta}\}$.

c. The formula (1) can be expressed using ζ only, through the equality:

$$\ln(\pi) - \frac{1}{2}\left(\Psi\left(\frac{1}{4}-\frac{s}{2}\right)+\Psi\left(\frac{1}{4}+\frac{s}{2}\right)\right) - \frac{\zeta'}{\zeta}\left(\frac{1}{2}+s\right) - \frac{\zeta'}{\zeta}\left(\frac{1}{2}-s\right) = 0. \qquad (2)$$

Corollary 1. *Under the additional assumptions that f is holomorphic on a neighborhood of the half-plane $\{\Im(s) \leq 0\}$ and that $\frac{f(\pm s)}{2^s}$ is integrable up to $+\infty$, the formula (1) becomes:*

$$\begin{aligned} &2\pi \sum_{0<b_k<n} f(-ib_k) + 2\pi \sum_{0<d_k<n} f\left(\frac{1}{2} - c_k - id_k\right) = \\ &\frac{\ln(\pi)}{i} \int_0^{-in} f(s)\,\mathrm{d}s - \frac{1}{2i} \int_0^{-in} f(s) \left(\Psi\left(\frac{1}{4} - \frac{s}{2}\right) + \Psi\left(\frac{1}{4} + \frac{s}{2}\right)\right) \mathrm{d}s \\ &- \frac{1}{i}\left(\int_{d_{-n,r}} f(s) \frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right) \mathrm{d}s - \int_{d_{0_-,r}} f(s) \frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right) \mathrm{d}s\right) \\ &- \frac{1}{i}\left(\int_{d_{-n,l}} f(s) \frac{\zeta'}{\zeta}\left(\frac{1}{2} - s\right) \mathrm{d}s - \int_{d_{0_-,l}} f(s) \frac{\zeta'}{\zeta}\left(\frac{1}{2} - s\right) \mathrm{d}s\right), \end{aligned} \tag{3}$$

where Ψ is the digamma function, ζ is the Riemann zeta function, $\int_0^{-in}$ means the integral along the oriented segment $[0, -in]$ and where the paths of integration $d_{-n,r}$ (respectively $d_{0_-,r}, d_{-n,l}, d_{0_-,l}$) are the horizontal half-lines starting from $+\infty - in$ (respectively $+\infty, -\infty - in, -\infty$) up to $-in$ (respectively 0), with a small semicircle below $\frac{1}{2}$ (resp. $-\frac{1}{2}$) for $d_{0_-,r}$ (resp. $d_{0_-,l}$).

Proof of the corollary 1. We make A tend to infinity in equality (1).

We also find the known result :

Corollary 2. *If $N(n)$ is the number of zeros of the zeta function whose imaginary part is between 0 and n, then we have:*

$$N(n) = 1 + \frac{1}{\pi} \arg\left(\pi^{-in/2} \Gamma\left(\frac{1}{4} + \frac{in}{2}\right)\right) + \frac{1}{\pi} \arg\left(\zeta\left(\frac{1}{2} + in\right)\right). \tag{4}$$

Proof of the corollary 2. Applying the formula (3) to the constant function $f = 1$, we obtain:

$$\begin{aligned} 2\pi N(n) = &-n\ln(\pi) + 2\Im\left(\ln\left(\Gamma\left(\frac{1}{4} + \frac{in}{2}\right)\right)\right) \\ &+ 2\Im\left(\ln\left(\zeta\left(\frac{1}{2} + in\right)\right)\right) - 2\Im\left(\ln\left(\zeta\left(\frac{1}{2} + i0_+\right)\right)\right). \end{aligned} \tag{5}$$

and the result follows, through the equality $\Im\left(\ln\left(\zeta\left(\frac{1}{2} + i0_+\right)\right)\right) = -\pi$.

Corollary 3. *If there exists a unique zero $\frac{1}{2} + ib_{k(n)}$ of multiplicity 1 (respectively, if there is no zero) such that $n < b_{k(n)} < n + 1$, the value of $b_{k(n)}$ (resp. 0) is given by:*

$$\begin{aligned} b_{k(n)} = &-\frac{1}{2\pi}\left(\left(n + \frac{1}{2}\right)\ln(\pi) + \Re\left(\int_{-in}^{-i(n+1)} s\Psi\left(\frac{1}{4} + \frac{s}{2}\right) \mathrm{d}s\right)\right) \\ &- \frac{1}{\pi}\Re\left(\int_{R^*_{n+1,A} - R^*_{n,A}} s\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right) \mathrm{d}s\right). \end{aligned} \tag{6}$$

Proof of the corollary 3. We apply the formula (3) to the function $f(s) = -is$.

We give two proofs of the theorem. The second, using the Mellin transform, is longer and the result is weaker (it must be assumed that the function f is entire), but this proof is more natural since it provides the formula.

I. First proof of the theorem

If we apply the residue formula to the integral $\int_0^{-in} f(s)\left(\ln(\pi) - \frac{1}{2}\left(\Psi\left(\frac{1}{4} - \frac{s}{2}\right) + \Psi\left(\frac{1}{4} + \frac{s}{2}\right)\right)\right) \mathrm{d}s$, then formula (??) can be written:

$$\begin{aligned}
&2\pi \sum_{0<b_k<n} f(-ib_k) + 2\pi \sum_{0<d_k<n} f\left(\frac{1}{2} - c_k - id_k\right) = \\
&\frac{1}{i}\int_{R^*_{n,A_1}} f(s)\left(\ln(\pi) - \frac{1}{2}\left(\Psi\left(\frac{1}{4} - \frac{s}{2}\right) + \Psi\left(\frac{1}{4} + \frac{s}{2}\right)\right)\right) \mathrm{d}s \\
&-\frac{1}{i}\left(\int_{R^*_{n,A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right)\mathrm{d}s + \int_{R^*_{n,-A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} - s\right)\mathrm{d}s\right),
\end{aligned} \tag{7}$$

where $\frac{1}{2} < A_1 < \frac{5}{2}$. We use the formula (2), and the previous formula becomes:

$$\begin{aligned}
&2\pi \sum_{0<b_k<n} f(-ib_k) + 2\pi \sum_{0<d_k<n} f\left(\frac{1}{2} - c_k - id_k\right) \\
=\ &\frac{1}{i}\int_{R^*_{n,A_1}} f(s)\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right) + \frac{\zeta'}{\zeta}\left(\frac{1}{2} - s\right)\right)\mathrm{d}s \\
&-\frac{1}{i}\left(\int_{R^*_{n,A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right)\mathrm{d}s + \int_{R^*_{n,-A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} - s\right)\mathrm{d}s\right) \\
=\ &\frac{1}{i}\int_{R^*_{n,A_1} - R^*_{n,A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right)\mathrm{d}s + \frac{1}{i}\int_{R^*_{n,A_1} - R^*_{n,-A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} - s\right)\mathrm{d}s,
\end{aligned} \tag{8}$$

which completes the proof of Lemma I.0, since, through the residue formula, we obtain:

$$\frac{1}{i}\int_{R^*_{n,A_1} - R^*_{n,A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} + s\right)\mathrm{d}s = 0 \tag{9}$$

and

$$2\pi \sum_{0<b_k<n} f(-ib_k) + 2\pi \sum_{0<d_k<n} f\left(\frac{1}{2} - c_k - id_k\right) = \frac{1}{i}\int_{R^*_{n,A_1} - R^*_{n,-A}} f(s)\frac{\zeta'}{\zeta}\left(\frac{1}{2} - s\right)\mathrm{d}s. \tag{10}$$

II. Second proof of the theorem

We need two lemmas.

Lemma II.1. *If u and n are two positive real numbers, if $\left(\frac{1}{2}+ib_k\right)_{k\in\mathbb{Z}-\{0\}}$ is the sequence of the zeros of the zeta function having real part $\frac{1}{2}$, if $(c_k+id_k)_{k\in\mathbb{Z}-\{0\}}$ is the sequence of the remaining non-trivial zeros, then we have:*

$$\begin{aligned}&2\pi\sum_{0<b_k<n}u^{-ib_k}+2\pi\sum_{0<d_k<n}u^{\frac{1}{2}-c_k-id_k}=\\&\frac{\ln(\pi)}{i}\int_0^{-in}u^s\mathrm{d}s-\frac{1}{2i}\int_0^{-in}u^s\left(\Psi\left(\frac{5}{4}-\frac{s}{2}\right)+\Psi\left(\frac{5}{4}+\frac{s}{2}\right)\right)\mathrm{d}s\\&+M_d^{-1}\left(\frac{\zeta'}{\zeta}+\frac{1}{s-1}\right)\left(-\frac{1}{i}\int_0^{-in}u^s\left(t^{s+\frac{1}{2}}+t^{-s+\frac{1}{2}}\right)\mathrm{d}s\right),\end{aligned}\tag{11}$$

where $\int_0^{-in}$ refers to the integral along the oriented segment $[0,-in]$, $M_d^{-1}(\frac{\zeta'}{\zeta}+\frac{1}{s-1})$ denotes the inverse Mellin transform "turning right" of the operator $\frac{\zeta'}{\zeta}+\frac{1}{s-1}$ and with the following conventions: $b_k,d_k>0$ if $k>0$, $0<c_k<1$, $c_k\neq\frac{1}{2}$, $b_{-k}=-b_k$, $c_{-k}+id_{-k}=1-(c_k+id_k)$ if $k\in\mathbb{Z}-\{0\}$.

Proof of the lemma II.1. We start from the known formula:

$$\sum_{k\in\mathbb{Z}-\{0\}}\frac{1}{s-\left(\frac{1}{2}+ib_k\right)}+\sum_{k\in\mathbb{Z}-\{0\}}\frac{1}{s-(c_k+id_k)}=-c+\frac{1}{s-1}+\frac{1}{2}\left(\gamma+\Psi\left(1+\frac{s}{2}\right)\right)+\frac{\zeta'}{\zeta}(s),\tag{12}$$

where $c=\frac{\gamma+\ln(\pi)}{2}$, γ is the Euler's constant and where the infinite sums have to be understood in the sense of the limits of symmetric partial sums.

By inverse Mellin transform "turning right"(see appendix), we obtain the formula:

$$\sum_{k\in\mathbb{Z}-\{0\}}[t^{-\frac{1}{2}-ib_k}1_{]\leftarrow,1]}]+\sum_{k\in\mathbb{Z}-\{0\}}[t^{-c_k-id_k}1_{]\leftarrow,1]}]=-c\delta_1+T+[t^{-1}1_{]\leftarrow,1]}]-\sum_{k\in\mathbb{N}-\{0\}}\Lambda(k)\delta_{\frac{1}{k}},\tag{13}$$

where the notation $[t^a1_{]\leftarrow,1]}]$ denotes the inverse Mellin transform "turning right" of the operator $\frac{1}{s+a}$, δ_a is the Dirac distribution at the point a, T is the operator defined by $T(\phi)=-\int_0^1 t\frac{\phi(t)-\phi(1)}{1-t^2}\mathrm{d}t$ (inverse "turning right" of the operator $\frac{1}{2}(\gamma+\Psi(1+\frac{s}{2}))$) and $-\sum_{k\in\mathbb{N}-\{0\}}\Lambda(k)\delta_{\frac{1}{k}}$ (where Λ is the von Mangoldt function) is the inverse Mellin transform "turning right" of the operator $\frac{\zeta'}{\zeta}$.

We now consider the function $\phi_{n,u}(t)=\sqrt{t}(1+(\delta_{\frac{1}{t}})h_u(\frac{1-t^{-in}}{i\ln(t)})$, where $\sqrt{t}$ is the operator of multiplication by $\sqrt{t}$, $\delta_{\frac{1}{t}}$ is defined by $(\delta_{\frac{1}{t}}(\phi))(t)=\phi(\frac{1}{t})$, u and n are positive real numbers and h_u is the homothety operator of ratio u, defined by $(h_u(\phi))(t)=\phi(ut)$.

Let A be a positive real number. By using the following equality:

$$\frac{1-t^{-in}}{i\ln(t)}=-\frac{1}{i}\int_0^{-in}t^s\mathrm{d}s=-\frac{1}{i}\int_{R_{n,A}}t^s\mathrm{d}s=-\frac{1}{i}\int_{R_{n,-A}}t^s\mathrm{d}s,\tag{14}$$

where $R_{n,A}$ (resp. $R_{n,-A}$) is the portion of the rectangle successively joining the points 0, A (resp. $-A$), $A-in$ (resp. $-A-in$), $-in$, we obtain:

$$\begin{aligned}&\phi_{n,u}(t) = \\ &-\frac{1}{i}\sqrt{t}\int_0^{-in} u^s\left(t^s+t^{-s}\right)\mathrm{d}s = -\frac{1}{i}\sqrt{t}\int_{R_{n,A}} u^s\left(t^s+t^{-s}\right)\mathrm{d}s = -\frac{1}{i}\sqrt{t}\int_{R_{n,-A}} u^s\left(t^s+t^{-s}\right)\mathrm{d}s.\end{aligned} \tag{15}$$

We calculate the image of $\phi_{n,u}$ by each operator of the formula (13).

If $k>0$:

$$\begin{aligned}&\left[t^{-c_k-id_k}1_{]\leftarrow,1]}\right](\phi_{n,u}) \\ = \;& -\frac{1}{i}\int_{R_{n,A}} u^s[t^{-c_k-id_k}1_{]\leftarrow,1]}]\left(t^{s+\frac{1}{2}}\right)\mathrm{d}s - \frac{1}{i}\int_{R_{n,-A}} u^s[t^{-c_k-id_k}1_{]\leftarrow,1]}]\left(t^{-s+\frac{1}{2}}\right)\mathrm{d}s \\ = \;& -\frac{1}{i}\int_{R_{n,A}} u^s\left(M\left([t^{-c_k-id_k}1_{]\leftarrow,1]}]\right)\left(s+\frac{1}{2}\right)\right)\mathrm{d}s \\ & -\frac{1}{i}\int_{R_{n,-A}} u^s\left(M\left([t^{-c_k-id_k}1_{]\leftarrow,1]}]\right)\left(-s+\frac{1}{2}\right)\right)\mathrm{d}s \\ = \;& -\frac{1}{i}\int_{R_{n,A}} u^s\left(\frac{1}{s+\frac{1}{2}-c_k-id_k}\right)\mathrm{d}s - \frac{1}{i}\int_{R_{n,-A}} u^s\left(\frac{1}{-s+\frac{1}{2}-c_k-id_k}\right)\mathrm{d}s\end{aligned} \tag{16}$$

and

$$\begin{aligned}&[t^{-c_{-k}-id_{-k}}1_{]\leftarrow,1]}](\phi_{n,u}) \\ = \;& -\frac{1}{i}\int_{R_{n,A}} u^s\left(\frac{1}{s+\frac{1}{2}-c_{-k}-id_{-k}}\right)\mathrm{d}s - \frac{1}{i}\int_{R_{n,-A}} u^s\left(\frac{1}{-s+\frac{1}{2}-c_{-k}-id_{-k}}\right)\mathrm{d}s \\ = \;& -\frac{1}{i}\int_{R_{n,A}} u^s\left(\frac{1}{s-\frac{1}{2}+c_k+id_k}\right)\mathrm{d}s - \frac{1}{i}\int_{R_{n,-A}} u^s\left(\frac{1}{-s-\frac{1}{2}+c_k+id_k}\right)\mathrm{d}s,\end{aligned} \tag{17}$$

therefore:

$$\begin{aligned}&\left([t^{-c_k-id_k}1_{]\leftarrow,1]}]+[t^{-c_{-k}-id_{-k}}1_{]\leftarrow,1]}]\right)(\phi_{n,u}) \\ = \;& -\frac{1}{i}\int_{R_{n,A}} u^s\left(\frac{1}{s+\frac{1}{2}-c_k-id_k}\right)\mathrm{d}s + \frac{1}{i}\int_{R_{n,-A}} u^s\left(\frac{1}{s+\frac{1}{2}-c_k-id_k}\right)\mathrm{d}s \\ & -\frac{1}{i}\int_{R_{n,A}} u^s\left(\frac{1}{s-\frac{1}{2}+c_k+id_k}\right)\mathrm{d}s + \frac{1}{i}\int_{R_{n,-A}} u^s\left(\frac{1}{s-\frac{1}{2}+c_k+id_k}\right)\mathrm{d}s \\ = \;& 2\pi\sum_{0<d_k<n} u^{\frac{1}{2}-c_k-id_k},\end{aligned} \tag{18}$$

if $A>\frac{1}{2}$.

By making $c_k=\frac{1}{2}$ and $d_k=b_k$ in formula (18), we obtain:

$$\left([t^{-\frac{1}{2}-ib_k}1_{]\leftarrow,1]}]+[t^{-\frac{1}{2}-ib_{-k}}1_{]\leftarrow,1]}]\right)(\phi_{n,u}) = 2\pi\sum_{0<b_k<n} u^{-ib_k}. \tag{19}$$

On the other hand:

$$-c\delta_1(\phi_{n,u}) = \frac{2c}{i}\int_0^{-in} u^s\mathrm{d}s = -2c\frac{1-u^{-in}}{i\ln(u)}, \tag{20}$$

and

$$
\begin{aligned}
& T(\phi_{n,u}) \\
= \;& -\frac{1}{i}\int_0^{-in} u^s \left(M\left(T\right)\left(s+\frac{1}{2}\right) + M\left(T\right)\left(-s+\frac{1}{2}\right)\right) \mathrm{d}s \\
= \;& -\frac{1}{2i}\int_0^{-in} u^s \left(2\gamma + \Psi\left(\frac{5}{4}+\frac{s}{2}\right) + \Psi\left(\frac{5}{4}-\frac{s}{2}\right)\right) \mathrm{d}s.
\end{aligned}
\tag{21}
$$

So, by using the equality $-\gamma + 2c = \ln(\pi)$, we obtain:

$$
\begin{aligned}
& 2\pi \sum_{0<b_k<n} u^{-ib_k} + 2\pi \sum_{0<d_k<n} u^{\frac{1}{2}-c_k-id_k} = \\
& \frac{\ln(\pi)}{i}\int_0^{-in} u^s \mathrm{d}s - \frac{1}{2i}\int_0^{-in} u^s \left(\Psi\left(\frac{5}{4}-\frac{s}{2}\right) + \Psi\left(\frac{5}{4}+\frac{s}{2}\right)\right) \mathrm{d}s \\
& + M_d^{-1}\left(\frac{\zeta'}{\zeta} + \frac{1}{s-1}\right)\left(-\frac{1}{i}\int_0^{-in} u^s \left(t^{s+\frac{1}{2}} + t^{-s+\frac{1}{2}}\right) \mathrm{d}s\right),
\end{aligned}
\tag{22}
$$

which completes the proof of Lemma II.1.

Proof of the lemma II.1.

Lemma II.2. *If u, n and A are positive real numbers, if $\left(\frac{1}{2}+ib_k\right)_{k\in\mathbb{Z}-\{0\}}$ is the sequence of the zeros of the zeta function having real part $\frac{1}{2}$, we have:*

$$
\begin{aligned}
& 2\pi \sum_{0<b_k<n} u^{-ib_k} + 2\pi \sum_{0<d_k<n} u^{\frac{1}{2}-c_k-id_k} = \\
& \frac{\ln(\pi)}{i}\int_0^{-in} u^s \mathrm{d}s - \frac{1}{2i}\int_0^{-in} u^s \left(\Psi\left(\frac{1}{4}-\frac{s}{2}\right) + \Psi\left(\frac{1}{4}+\frac{s}{2}\right)\right) \mathrm{d}s \\
& -\frac{1}{i}\int_{R^*_{n,A}} u^s \frac{\zeta'}{\zeta}\left(s+\frac{1}{2}\right) \mathrm{d}s - \frac{1}{i}\int_{R^*_{n,-A}} u^s \frac{\zeta'}{\zeta}\left(-s+\frac{1}{2}\right) \mathrm{d}s,
\end{aligned}
\tag{23}
$$

where the path of integration $R^*_{n,A}$ (resp. $R^*_{n,-A}$) is the portion of the rectangle successively joining the points 0, A (resp. $-A$), $A-in$ (resp. $-A-in$), $-in$ (with a small semicircle below $\frac{1}{2}$ (resp. $-\frac{1}{2}$) if $A>\frac{1}{2}$) and with the following conventions: $b_k>0$ if $k>0$, $b_{-k}=-b_k$ if $k\in\mathbb{Z}-\{0\}$.

Proof of the lemma II.2.

$$
\begin{aligned}
& M_d^{-1}\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right)\left(-\frac{1}{i}\int_0^{-in} u^s \left(t^{s+\frac{1}{2}} + t^{-s+\frac{1}{2}}\right) \mathrm{d}s\right) \\
= \;& M_d^{-1}\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right)\left(-\frac{1}{i}\int_{R_{n,A}} u^s t^{s+\frac{1}{2}} \mathrm{d}s - \frac{1}{i}\int_{R_{n,-A}} u^s t^{-s+\frac{1}{2}} \mathrm{d}s\right) \\
= \;& -\frac{1}{i}\int_{R_{n,A}} u^s \left(\frac{\zeta'}{\zeta}\left(s+\frac{1}{2}\right) + \frac{1}{s-\frac{1}{2}}\right) \mathrm{d}s - \frac{1}{i}\int_{R_{n,-A}} u^s \left(\frac{\zeta'}{\zeta}\left(-s+\frac{1}{2}\right) - \frac{1}{s+\frac{1}{2}}\right) \mathrm{d}s.
\end{aligned}
\tag{24}
$$

We modify (if $A > \frac{1}{2}$) the integration path $R_{n,A}$ (resp. $R_{n,-A}$), avoiding the point $\frac{1}{2}$ (resp. $-\frac{1}{2}$) by a small semicircle below, and we call $R^*_{n,A}$ (resp. $R^*_{n,-A}$) the new path resulting. We have:

$$
\begin{aligned}
&M_d^{-1}\left(\frac{\zeta'}{\zeta}(s)+\frac{1}{s-1}\right)\left(-\frac{1}{i}\int_0^{-in} u^s\left(t^{s+\frac{1}{2}}+t^{-s+\frac{1}{2}}\right)\mathrm{d}s\right)\\
= \ &-\frac{1}{i}\int_{R^*_{n,A}} u^s\left(\frac{\zeta'}{\zeta}\left(s+\frac{1}{2}\right)+\frac{1}{s-\frac{1}{2}}\right)\mathrm{d}s-\frac{1}{i}\int_{R^*_{n,-A}} u^s\left(\frac{\zeta'}{\zeta}\left(-s+\frac{1}{2}\right)-\frac{1}{s+\frac{1}{2}}\right)\mathrm{d}s\\
= \ &-\frac{1}{i}\int_{R^*_{n,A}} u^s\frac{\zeta'}{\zeta}\left(s+\frac{1}{2}\right)\mathrm{d}s-\frac{1}{i}\int_{R^*_{n,-A}} u^s\frac{\zeta'}{\zeta}\left(-s+\frac{1}{2}\right)\mathrm{d}s+\frac{1}{i}\int_0^{-in} u^s\left(\frac{1}{-s+\frac{1}{2}}+\frac{1}{s+\frac{1}{2}}\right)\mathrm{d}s,
\end{aligned}
\tag{25}
$$

and, by using Lemmas II.1, **??**, and the following equality:

$$
\begin{aligned}
&-\frac{1}{2i}\int_0^{-in} u^s\left(\Psi\left(\frac{5}{4}+\frac{s}{2}\right)+\Psi\left(\frac{5}{4}-\frac{s}{2}\right)\right)\mathrm{d}s=\\
&-\frac{1}{2i}\int_0^{-in} u^s\left(\Psi\left(\frac{1}{4}+\frac{s}{2}\right)+\Psi\left(\frac{1}{4}-\frac{s}{2}\right)\right)\mathrm{d}s-\frac{1}{i}\int_0^{-in} u^s\left(\frac{1}{s+\frac{1}{2}}+\frac{1}{-s+\frac{1}{2}}\right)\mathrm{d}s,
\end{aligned}
\tag{26}
$$

we obtain:

$$
\begin{aligned}
&2\pi\sum_{0<b_k<n} u^{-ib_k}=\\
&\frac{\ln(\pi)}{i}\int_0^{-in} u^s\mathrm{d}s-\frac{1}{2i}\int_0^{-in} u^s\left(\Psi\left(\frac{1}{4}-\frac{s}{2}\right)+\Psi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\mathrm{d}s\\
&-\frac{1}{i}\int_{R^*_{n,A}} u^s\frac{\zeta'}{\zeta}\left(s+\frac{1}{2}\right)\mathrm{d}s-\frac{1}{i}\int_{R^*_{n,-A}} u^s\frac{\zeta'}{\zeta}\left(-s+\frac{1}{2}\right)\mathrm{d}s,
\end{aligned}
\tag{27}
$$

which completes the proof of Lemma II.2.

End of the proof of the theorem. We make the additional assumption that f is an entire function. We set $f(s)=\sum_{l=0}^{+\infty} f_l s^l$ and $\theta = u\frac{\mathrm{d}}{\mathrm{d}u}$.

By applying the derivation of order l ($l \in \mathbb{N}$) θ^l in formula (13), we obtain:

$$
\begin{aligned}
&2\pi\sum_{0<b_k<n}(-ib_k)^l u^{-ib_k}+2\pi\sum_{0<d_k<n}\left(\frac{1}{2}-c_k-id_k\right)^l u^{\frac{1}{2}-c_k-id_k}=\\
&\frac{\ln(\pi)}{i}\int_0^{-in} s^l u^s\mathrm{d}s-\frac{1}{2i}\int_0^{-in} s^l u^s\left(\Psi\left(\frac{1}{4}-\frac{s}{2}\right)+\Psi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\mathrm{d}s\\
&-\frac{1}{i}\int_{R^*_{n,A}} s^l u^s\frac{\zeta'}{\zeta}\left(s+\frac{1}{2}\right)\mathrm{d}s-\frac{1}{i}\int_{R^*_{n,-A}} s^l u^s\frac{\zeta'}{\zeta}\left(-s+\frac{1}{2}\right)\mathrm{d}s.
\end{aligned}
\tag{28}
$$

Summing over l the relation (23) multiplied by f_l, we have:

$$
\begin{aligned}
&2\pi\sum_{0<b_k<n} f(-ib_k)\,u^{-ib_k}=\\
&\frac{\ln(\pi)}{i}\int_0^{-in} f(s)\,u^s\mathrm{d}s-\frac{1}{2i}\int_0^{-in} f(s)\,u^s\left(\Psi\left(\frac{1}{4}-\frac{s}{2}\right)+\Psi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\mathrm{d}s\\
&-\frac{1}{i}\int_{R^*_{n,A}} f(s)\,u^s\frac{\zeta'}{\zeta}\left(s+\frac{1}{2}\right)\mathrm{d}s-\frac{1}{i}\int_{R^*_{n,-A}} f(s)\,u^s\frac{\zeta'}{\zeta}\left(-s+\frac{1}{2}\right)\mathrm{d}s,
\end{aligned}
\tag{29}
$$

and, by making $u = 1$:

$$
\begin{aligned}
&2\pi \sum_{0<b_k<n} f(-ib_k) = \\
&\frac{\ln(\pi)}{i} \int_0^{-in} f(s)\,\mathrm{d}s - \frac{1}{2i} \int_0^{-in} f(s) \left(\Psi\left(\frac{1}{4} - \frac{s}{2}\right) + \Psi\left(\frac{1}{4} + \frac{s}{2}\right)\right) \mathrm{d}s \\
&- \frac{1}{i} \int_{R^*_{n,A}} f(s) \frac{\zeta'}{\zeta} \left(s + \frac{1}{2}\right) \mathrm{d}s - \frac{1}{i} \int_{R^*_{n,-A}} f(s) \frac{\zeta'}{\zeta} \left(-s + \frac{1}{2}\right) \mathrm{d}s,
\end{aligned} \tag{30}
$$

which completes the second proof of the theorem.

Remark. If we use the inverse "turning left" M_g^{-1}, we obtain the formula with $A < 0$.

Appendix

Some concepts used here were developed in [1].

A.1 Some notations

In the following, we always identify a function q with the operator of multiplication by q: $\phi \to q\phi$.

The variable t will always belong to the universal cover of $\mathbb{C} - \{0\}$ and the variable s generally to the set $\mathbb{C}$ of the complex numbers.

We say that a linear operator is "in t" (resp. "in s") if it is defined on, and takes values in sets of functions of the variable t (resp. s). An example of operator "in t" is the operator Q (locally) defined by:

$$Q\left(\sum_{j=0}^{+\infty} \phi_j \left(\ln(t)\right)^j\right) = \sum_{k \geq j} Q_{(k,j)} \phi_j \left(\ln(t)\right)^k, \tag{31}$$

where $Q = \left(Q_{(k,j)}\right)_{(k,j) \in \mathbb{N} \times \mathbb{N}}$ is an infinite matrix with complex coefficients satisfying $Q_{(k,j)} = 0$ for $k - j \leq 0$ (with suitable assumptions on the coefficients $Q_{(k,j)}$ and ϕ_j to ensure the convergence of the series).

$h_\lambda (\lambda \in \mathbb{C})$ is the homothety operator: $\phi(t) \to \phi(\lambda t)$.

If g is a function, δ_g is the operator defined by: $\delta_g(\phi) = \phi \circ g$.

The notation $[g]$ denotes the operator (distribution) associated with the function g by the formula: $[g](\phi) = \int_0^{+\infty} g(t)\, \phi(t) \frac{\mathrm{d}t}{t}$.

$1_{[a,b]}$ is the indicator function of the interval $[a, b]$.

θ is the derivation $t\frac{\mathrm{d}}{\mathrm{d}t}$.

The derivative $\theta(Q)$ of the linear operator Q is defined by the Lie bracket: $\theta(Q) = [\theta, Q]$.

$M(Q)$ is the Mellin transform of the linear operator Q.

A.2 Some definitions

The Mellin transform of a linear operator "in t" is a linear operator "in s".
If $q(t) = \sum_{j=0}^{+\infty} q_j \left(\ln(t)\right)^j$ and $\phi(s) = \sum_{j=0}^{+\infty} \phi_j \left(s\right)^j$ the Mellin transform of the operator q is (locally) defined by:

$$M(q)(\phi) = \sum_{j=0}^{+\infty} (-1)^j j! q_j \phi_j. \tag{32}$$

Example. $M(t^a) = \delta_{-a}$.

More generally, if Q is a linear operator:

$$\left(M(Q)(\phi)\right)(s) = M\left(Q\left(t \to t^s\right)\right)\left(\sigma \to \phi(\sigma + s)\right). \tag{33}$$

In particular, if T is a linear operator with values in the constant functions (i.e. a distribution), $M(T)$ is the function (of s) $T(t^s)$ (i.e. the operator of multiplication by $T(t^s)$).

Examples. $M(\delta_a) = a^s$, $M([g])(s) = \int_0^{+\infty} g(t)\, t^s \frac{\mathrm{d}t}{t}$.

We also define two convolution products: if Q and R are linear operators:

$$((Q * R)(\phi))(s) = Q\,(u \to R\,(v \to \phi(u+v))\,(s-u))\,(s), \tag{34}$$

and

$$\left((Q *^\theta R)(\phi)\right)(t) = Q\left(u \to R\,(v \to \phi(uv))\left(\frac{t}{u}\right)\right)(t). \tag{35}$$

Examples. $\delta_0 * Q = Q * \delta_0 = Q$, $\delta_1 *^\theta Q = Q *^\theta \delta_1 = Q$, and, if f and g are functions: $f * g = fg(0)$, $f *^\theta g = fg(1)$, $(fQ) * R = f\,(Q * R)$, $(fQ) *^\theta R = f\left(Q *^\theta R\right)$, $(Q * f)(s) = Q\,(u \to f(s-u))\,(s)$, $\left(Q *^\theta f\right)(t) = Q\left(u \to f(\frac{t}{u})\right)(t)$.

A.3 Some general formulas

$$\begin{array}{lclcl} M\left(Q *^\theta R\right) & = & M(R) \circ M(Q) & & \\ M(Q \circ R) & = & M(R) * M(Q) & & \\ M(\theta(Q)) & = & M(Q) \circ (-s) & & \\ \theta\left(Q *^\theta R\right) & = & \theta(Q) *^\theta R & = & Q *^\theta \theta(R) + (\theta \circ Q) *^\theta R \end{array}$$

A.4 A list of Mellin transforms

$M(t^a) = \delta_{-a}$		
$M(t^a \delta_1) = \tau_a$		where τ_a is the translation operator: $\phi(s) \to \phi(s-a)$
$M(\delta_a) = a^s$		
$M(h_\lambda) = \lambda^s \delta_0$		
$M(\delta_{t^a}) = h_{1-a}$		
$M([g])(s) = \int_0^{+\infty} g(t)\, t^s \frac{\mathrm{d}t}{t}$		
$M\left([t^{-a} 1_{[0,1]}]\right)(s) = \frac{1}{s-a}$	for $\Re(s) > \Re(a)$	$[t^{-a} 1_{[0,1]}]$ is also denoted by $M_d^{-1}\left(\frac{1}{s-a}\right)$, M_d^{-1} is the inverse "turning right"
$M\left([t^{-a} 1_{[1,+\infty[}]\right)(s) = -\frac{1}{s-a}$	for $\Re(s) < \Re(a)$	$[t^{-a} 1_{[1,+\infty[}]$ is also denoted by $M_g^{-1}\left(-\frac{1}{s-a}\right)$, M_g^{-1} is the inverse "turning left"
$M\left(I^k\right) = \left(\frac{1}{s}\right)^k \delta_0$	for $\Re(s) > 0$	where I is the operator: $\phi \to \int_0^t \phi(u) \frac{\mathrm{d}u}{u}$
$M\left(\tilde{I}^k\right) = \left(\frac{1}{s}\right)^k \delta_0$	for $\Re(s) < 0$	where $\tilde{I}$ is the operator: $\phi \to \int_{+\infty}^t \phi(u) \frac{\mathrm{d}u}{u}$
$M\left(\theta^k\right) = (s)^k \delta_0$		

$M\left(\delta_1 \sum_{k=1}^{+\infty} a^{k-1} I^k\right) = \frac{1}{s-a}$	for $\lvert s\rvert > \lvert a\rvert$ and $\Re(s) > 0$	$\delta_1 \sum_{k=1}^{+\infty} a^{k-1} I^k$ is also denoted by $M_d^{-1}\left(\frac{1}{s-a}\right)$
$M\left(\delta_1 \sum_{k=1}^{+\infty} a^{k-1} \tilde{I}^k\right) = \frac{1}{s-a}$	for $\lvert s\rvert > \lvert a\rvert$ and $\Re(s) < 0$	$\delta_1 \sum_{k=1}^{+\infty} a^{k-1} \tilde{I}^k$ is also denoted by $M_g^{-1}\left(\frac{1}{s-a}\right)$
$M\left(-\delta_1 \sum_{k=0}^{+\infty} \left(\frac{1}{a}\right)^{k+1} \theta^k\right) = \frac{1}{s-a}$	for $\lvert s\rvert < \lvert a\rvert$ and $a \neq 0$	$-\delta_1 \sum_{k=0}^{+\infty} \left(\frac{1}{a}\right)^{k+1} \theta^k$ is also denoted by $M_g^{-1}\left(\frac{1}{s-a}\right)$ (resp. $M_d^{-1}\left(\frac{1}{s-a}\right)$) if $a > 0$ (resp. $a < 0$)
$M\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right) = s(s-1)...(s-k+1)\delta_k$		
$M(T)(s) = \frac{1}{2}(\gamma + \Psi(1 + \frac{s}{2}))$	for $\Re(s) > -2$	where T is the operator: $\phi \to -\int_0^1 t\dfrac{\phi(t) - \phi(1)}{1 - t^2}\mathrm{d}t$
$M\left(-\sum_{k\in\mathbb{N}-\{0\}} \Lambda(k)\delta_{\frac{1}{k}}\right)(s) = \frac{\zeta'}{\zeta}(s)$	for $\Re(s) > 1$	where Λ is the von Mangoldt function

A.5 Analytic continuation of an operator "in t"

Example. The operator ("in t") $[t^{-a}1_{[0,1]}]$ (resp. $[t^{-a}1_{[1,+\infty[}]$) extends analytically to an operator denoted $[t^{-a}1_{]\leftarrow,1]}]$ or $M_d^{-1}\left(\frac{1}{s-a}\right)$ (resp. $[t^{-a}1_{[1,\rightarrow[}]$ or $M_g^{-1}\left(\frac{1}{s-a}\right)$) (inverse "turning right" (resp. "turning left") of $\frac{1}{s-a}$), for s belonging to the universal cover of $\mathbb{C} - \{a\}$, by local inversion.

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References

[1] H. Charrière, *Opérateurs linéaires analytiques. Transformée de Fourier. Produit de convolution. Applications.*, Kumamoto Journal of Mathematics, **8** (1995), 1-49